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SOME ASYMPTOTIC EXPANSIONS OF THE DISPERSION RELATION FOR AN INCOMPRESSIBLE ELASTIC PLATE

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Abstract—This paper is concerned with a general asymptotic analysis of the dispersion relation associated with waves propagating in a pre-stressed, incompressible elastic plate. In the high wave number limit it is well-known that, whenever a real surface wave speed exists, the fundamental modes of both symmetric and anti-symmetric motions tend to this surface wave speed, with all harmonics tending to a single shear wave speed limit. The character of the two dispersion curves in the moderate and high wave number regimes falls into one of two distinct cases, these being dependent on pre-stress. In the first case all the harmonics are monotonic decreasing functions and as such the asymptotic analysis in this case offers a modest generalisation of an earlier study, see Rogerson and Fu (Rogerson, G. A. and Fu, Y. B. (1995) An asymptotic analysis of the dispersion relation of a pre-stressed incompressible elastic plate. *Acta Mechanica* **111**, 59–77). In contrast, the second case is quite different in character with the passage to the high wave number limit accompanied by sinusoidal behaviour. This behaviour is fully elucidated by obtaining asymptotic expansions which give phase speed as a function of wave number, pre-stress and harmonic number, sinusoidal terms being found to occur at third order. Both these asymptotic expansions and ones obtained for high harmonic number are found to provide excellent agreement with numerical solutions for Varga materials in the appropriate regimes. It is envisaged that the expansions derived in this paper may well find important potential applications in the numerical inversion of the transform solution sometimes used in impact problems. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The problem of waves propagating in an isotropic elastic plate is a classical elastodynamic problem. In recent years the effects of pre-stress on wave propagation in elastic plates has been a topic of increasingly active research, see, e.g., Rogerson and Fu (1995), Ogden and Roxburgh (1993) and Rogerson and Sandiford (1996). These papers are specifically concerned with the problems of wave transmission or vibration in incompressible elastic plates and laminates and contain reference to many of the early studies of this and related problems. Although it is certainly true that pre-stress may well be induced in modern composites during the manufacturing process, in this paper it is envisaged that pre-stress arises from the action of external forces. It is therefore tacitly assumed that the plate we later discuss is employed in some load supporting capacity. Indeed, one very specific motivation for all the recent studies, aimed at elucidating the effects of pre-stress, is born out of a desire to understand the mechanical properties of the rubber-like components used as vibration insulators in bridge supports and large building. These components have great potential in modern methods of earthquake protection, see Sheridan *et al.* (1992).

In this paper a general asymptotic analysis of the dispersion relation associated with small amplitude waves superimposed on a finite homogeneous deformation in an infinite, incompressible elastic plate is carried out. Such relations provide an implicit relationship between phase speed and wave number, consists of an infinite number of branches and are obtained by satisfying incremental traction free boundary conditions on the two free surfaces of the plate. As such this paper generalises an earlier study in which such asymptotic expansions were obtained for a limited class of incompressible material, see Rogerson and Fu (1995). It will be observed that distinct new asymptotic features arise in the more general

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treatment which could not possibly manifest themselves in this earlier study. An important motivation for carrying out such a detailed study is that a full understanding of the dispersion relation is a necessary pre-requisite in determining transient impact response. Indeed, several authors have addressed the impact problem by solving the full three dimensional equations of elasticity exactly, only resorting to numerical methods in the inversion of integral transforms. In using this method for line impact problems it is found that the denominator of the solution integrand is in fact the dispersion relation associated with the corresponding traction free boundary value problem, see, e.g., Rogerson (1992). The solution in terms of space and time variables is then obtained by summing the integral contributions from each branch of the dispersion relation, each point on a branch being a pole of the integrand.

We begin this paper in Section 2 with a brief review of the equations which govern small amplitude motions superimposed upon a large homogeneous deformation. Both the equations of motions and a measure of incremental traction are obtained for an incompressible elastic solid under the assumption of plane strain. It is observed in these equations that there are only three material constants α, β and γ, these being linear functions of the components of the elasticity tensor. The dispersion relation for waves propagating in an incompressible elastic plate is derived in the third section. This relation is decomposed into two components, one corresponding to extensional waves the other to flexural. In Section 4 the dispersion relations are solved numerically to give phase speed as a function of wave number in the case of a Varga material. It is observed that the character of these solutions is dependent on whether $2\beta \geqslant \alpha$ or $2\beta < \alpha$. In both cases the fundamental mode is the only one which retains a finite wave speed limit as $kh \to 0$, all harmonics having a corresponding infinite wave speed limit. As $kh \to \infty$ the limit in both cases is the Rayleigh surface wave speed. However, in the second case it is observed that the fundamental modes of both extensional and flexural waves cross over at various values of wave number. In the former case ($2\beta \geqslant \alpha$) the harmonics of both extensional and flexural motions interlace each other with each branch a monotonically decreasing function of wave number tending towards a certain limiting wave speed. In this case the results are manifestly the same as those previously obtained for a restricted class of incompressible material for which $2\beta = \alpha+\gamma$, see Rogerson and Fu (1995). The behaviour of the harmonics in the second case ($2\beta < \alpha$) is however significantly different, a point seemingly first noted in Ogden and Roxburgh (1993), each harmonic now exhibiting sinusoidal behaviour before it reaches a different short wavelength limit to case 1. In addition, each harmonic of extensional motion forms a pair with the corresponding branch for flexural motion, these pairs crossing over periodically as the wave number increases and after each has fallen below a certain critical phase speed.

Guided by the numerical solutions outlined above, complete asymptotic analyses of the dispersion relations for extensional wave and flexural waves are carried out in Sections 5 and 6, respectively. The asymptotic behaviour of the fundamental mode in each case is shown to be in accordance with the numerical indications already discussed. Asymptotic expansions are derived for the harmonics which give the phase speed as a function of wave number, harmonic number and pre-stress. In the case in which $2\beta \geqslant \alpha$ these expansions generalise previous results, see Rogerson and Fu (1995). However, as might be expected, when $2\beta < \alpha$ the asymptotic expansions are completely different in character. In this case the expansions for extensional and flexural motions are identical up to second order, whilst at third order a sinusoidal term occurs. Furthermore, this third order term for extensional waves is in general opposite in sign to that occurring in the corresponding solution for flexural waves, the third order terms vanishing at certain values of wave number. Each of the final two sections is concluded with the derivation of appropriate expansions for large harmonic number in the moderate wave number regime. The various asymptotic expansions are shown to provide excellent agreement with numerical results in the appropriate wave number regimes.

2. GOVERNING EQUATIONS

The equations which govern small amplitude time-dependent motions, superimposed on a large homogeneous deformation in an incompressible elastic solid, are briefly derived in this section. For further details of both these basic equations, and the subsequent derivation of the dispersion relations, the reader is referred to Rogerson and Fu (1995) and Ogden and Roxburgh (1993). These papers each concern aspects of the dynamic response of a pre-stressed, incompressible elastic plate. However, the latter contains no detailed asymptotic discussion of the dispersion relation, whilst the former only does so for a limited class of incompressible material. It will be seen in later sections of this present paper that interesting and fundamentally different asymptotic features arise in the general case which could not possibly manifest themselves in the previous restricted study.

We shall consider a homogeneous elastic body composed of a non-heat conducting, incompressible elastic material which possesses an initial unstressed state B_o. A purely homogeneous deformation is then imposed on B_o, the resulting finitely stressed equilibrium state being denoted by B_e. Finally, a small time-dependent motion is superimposed on B_e, with the final current configuration denoted by B_t. The position vectors of a representative particle are denoted by X_A, $x_i(X_A)$ and $\tilde{x}_i(X_A, t)$ in B_o, B_e and B_t, respectively. The deformation gradient associated with the total deformation $B_o \to B_t$ may now be expressed in the form

$$F_{iA} = (\delta_{ij} + u_{i,j})\bar{F}_{jA}, \tag{1}$$

in which a comma indicates differentiation with respect to the implied spatial coordinate component in B_e, an overbar evaluation in B_e and $u_i(X_A, t)$ a time dependent infinitesimal displacement associated with the secondary deformation $B_e \to B_t$. A common starting point for problems involving internal constraints is the introduction of a pseudo strain energy function. For incompressibility this function in its most general form is given by

$$W(\mathbf{F}) = W_o(\mathbf{F}) - p(J-1), \quad (J-1) \equiv 0, \quad J = \det \mathbf{F}. \tag{2}$$

In eqn $(2)_1$ $W_o(\mathbf{F})$ generates the constitutive part of the stress, whilst the additional term, constrained to be zero for all material deformations, generates a workless reaction stress. The scalar multiplier p plays the role of Lagrange multiplier, is interpreted as a hydro-static pressure and must ultimately be chosen so that all equations of motion and any prescribed boundary conditions are satisfied.

In the absence of body forces the equations of motion are

$$\pi_{iA,A} = \rho \ddot{u}_i, \quad \pi_{iA} = \frac{\partial W}{\partial F_{iA}}, \tag{3}$$

where π_{iA} is the first Piola-Kirchhoff stress, ρ is the density per unit volume of B_o and a superimposed dot indicates differentiation with respect to time. These equations of motion are now expressible in the form

$$\left(\frac{\partial W_o}{\partial F_{iA}} \bar{F}_{pA} - pF_{Ai}^{-1} \bar{F}_{pA}\right)_{,p} = \rho \ddot{u}_i, \tag{4}$$

use having been made of eqn $(2)_1$. Linearised equations of motion may now be obtained by expanding $\partial W_o / \partial F_{iA}$ as a Taylor series about $\mathbf{F} = \bar{\mathbf{F}}$. If this linearisation process is carried out, use made of (1) and p assumed to have a time dependent increment p^*, and therefore $p = \bar{p} + p^*$, we obtain the linearised equations of motion

$$B_{jilk} u_{k,lj} - p^*_{,i} = \rho \ddot{u}_i, \quad B_{ijkl} = \bar{F}_{iA} \bar{F}_{kC} \frac{\partial^2 W_o}{\partial F_{jA} \partial F_{lC}}(\bar{\mathbf{F}}), \tag{5}$$

in which B_{ijkl} is the fourth order elasticity tensor. Similarly, a scaled measure of incremental traction, motivated by the dead load condition, may also be obtained, thus

$$\tau_i = B_{jilk}u_{k,l}n_j + \bar{p}u_{j,i}n_j - p^*n_i, \tag{6}$$

in which $\mathbf{n}$ is the outward unit normal to a material surface in B_e.

3. THE DISPERSION RELATION

The problem of wave propagation in a plate of finite thickness $2h$, composed of a material characterised by the strain energy function $(2)_1$, is now addressed. An appropriate Cartesian coordinate system $Ox_1x_2x_3$ is chosen coincident with the principal axes of stress in B_e and orientated such that Ox_2 is normal to the plane of the plate, with origin O in the mid-plane. In order to simplify the governing equations a plane strain assumption is made in which u_1 and u_2 are independent of x_3 and $u_3 \equiv 0$. The linearised incompressibility condition may now be written in the form $u_{1,1} = -u_{2,2}$, with the two non-trivial equations of motion now given by

$$B_{j1lk}u_{k,lj} - p^*_{,1} = \rho\ddot{u}_1, \quad B_{j2lk}u_{k,lj} - p^*_{,2} = \rho\ddot{u}_2. \tag{7}$$

The two corresponding non-trivial incremental traction components are obtainable from (6) and enable us to obtain the traction free boundary conditions

$$B_{21lk}u_{k,l}\bar{p}u_{2,1} - p^*n_1 = B_{22lk}u_{k,l} - \bar{p}u_{2,2} - p^*n_2 = 0 \quad \text{at } x_2 = \pm h. \tag{8}$$

Travelling wave solutions of the equations of motion (7) are now sought in the form

$$(u_1, u_2, p^*) = (U, V, kP)e^{kqx_2}e^{ik(x_1 - vt)}, \tag{9}$$

within which k is the wave number and v the phase speed. If these solutions are inserted into (7) and the linearised incompressibility condition a system of linear homogeneous equations is obtained. A non-trivial solution of this system will exist provided

$$\gamma q^4 + (\rho v^2 - 2\beta)q^2 + (\alpha - \rho v^2) = 0, \tag{10}$$

in which

$$\alpha = B_{1212}, \quad 2\beta = B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221}, \quad \gamma = B_{2121}.$$

In obtaining (10) it is noted that all non-zero components of B_{ijkl} are of the form B_{iijj}, B_{ijji} or B_{ijij}, $(i,j \in \{1,2,3\})$, see Ogden (1984). If the two solutions from (10) are denoted by q_1^2 and q_2^2 it is observed that

$$q_1^2 + q_2^2 = \frac{2\beta - \rho v^2}{\gamma}, \quad q_1^2q_2^2 = \frac{\alpha - \rho v^2}{\gamma}. \tag{11}$$

Solutions for U, V and P may be represented as a linear combination of the four independent solutions obtained from (10). These solutions may then be used in conjunction with (8) to yield a system of four homogeneous equations. This system may be shown to have a non-trivial solution provided either

$$q_2\{\gamma(q_1^2+1) - \sigma_2\}^2 \tanh(kq_1h) = q_1\{\gamma(q_2^2+1) - \sigma_2\}^2 \tanh(kq_2h), \tag{12}$$

or

$$q_2\{\gamma(q_1^2+1) - \sigma_2\}^2 \tanh(kq_2h) = q_1\{\gamma(q_2^2+1) - \sigma_2\}^2 \tanh(kq_1h), \tag{13}$$

in which the relation $\bar{p} = B_{2121} - B_{2112} - \sigma_2$ has been used to eliminate $\bar{p}$ in favour of σ_2, the principal Cauchy stress along Ox_2 in B_e. Equations (12) and (13) together form the dispersion relation which will in general allow the phase speed to be obtained as a function of the scaled wave number kh. It is remarked that (12) corresponds to extensional waves, whilst (13) corresponds to flexural waves.

4. NUMERICAL SOLUTION OF THE DISPERSION RELATIONS

Detailed numerical investigations of (12) and (13) have previously been carried out, see Ogden and Roxburgh (1993). In this section, two distinctly different numerical solutions are presented which exemplify two distinct cases which may arise, thus providing a necessary basis for later asymptotic analysis. For all the numerical calculations within this section the constitutive part of the strain energy function is assumed to be the Varga strain energy

$$W_o(\mathbf{F}) = \mu(\lambda_1+\lambda_2+\lambda_3-3), \tag{14}$$

in which μ is a constant and λ_i the principal stretch along Ox_i. For this specific strain energy function it is readily established that

$$\alpha = \frac{\mu\lambda_1^2}{\lambda_1+\lambda_2}, \quad \gamma = \frac{\mu\lambda_2^2}{\lambda_1+\lambda_2}, \quad 2\beta = \frac{2\mu\lambda_1\lambda_2}{\lambda_1+\lambda_2}. \tag{15}$$

The two distinct cases already mentioned will now each be discussed in turn.

4.1. *Case* 1

In Fig. 1(a) the fundamental modes of both extensional and flexural waves are presented. It is evident from this figure that the fundamental mode of both (12) and (13) both tend to the same wave speed limit as $kh \to \infty$. This limit will be shown later to be the appropriate Rayleigh surface wave speed. On the other hand, as $kh \to 0$ the fundamental modes tend to distinct finite limits. For both extensional and flexural waves the harmonics may be observed in Fig. 1(b) to behave in a very similar way and are such that $\rho v^2 \to \infty$ as $kh \to 0$, and as $kh \to \infty$ all harmonics are approaching the same limit. In the passage to the limit $kh \to \infty$ numerical calculations indicate that for very large wave number one of (q_1, q_2) is purely imaginary with a modulus which tends to zero as $kh \to \infty$, whilst the other in general remains real and finite. Without loss of generality it is therefore assumed that $q_1 = i\hat{q}$, with $\hat{q} \geqslant 0$ and real, and it is then assumed that as $kh \to \infty$

$$\hat{q} \to 0, \quad \rho v^2 \to \alpha, \tag{16}$$

for all harmonics. Equation (10) may now be invoked to establish a condition which ensures that q_2 will remain real as $\rho v^2 \to \alpha$, allowing us to classify case 1 as follows

$$\text{Case 1}: 2\beta-\alpha \geqslant 0. \tag{17}$$

A point of note concerning this limit is that it corresponds to the speed of a shear wave propagating along Ox_1, see (9) with $q = 0$. The type of behaviour observed in Fig. 1 is manifestly the same as that previously investigated for a restricted class of incompressible elastic material for which $\alpha+\gamma = 2\beta$, see Rogerson and Fu (1995).

4.2. *Case* 2

In this case the behaviour of the two fundamental modes in Fig. 2(a) is at first glance manifestly the same as the previous case shown in Fig. 1(a), except that now the two curves intersect and cross over. Furthermore, numerical calculations indicate that these two fundamental modes continue to cross over as kh increases. A glance at Fig. 2(b) soon reveals that significant differences in the behaviour of the harmonics in all but the very low wave number regime. In this case the numerical calculations for the harmonics indicate

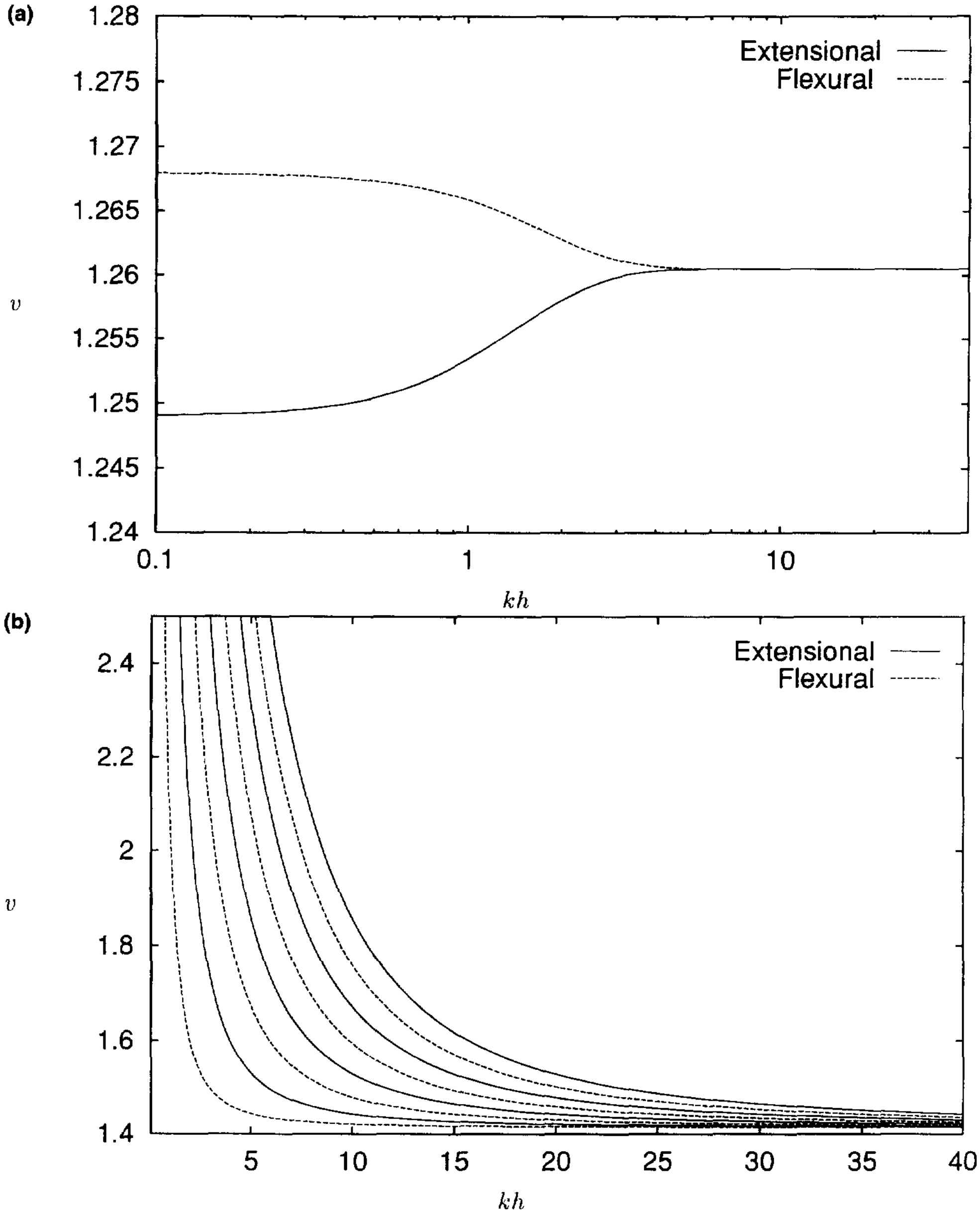

Fig. 1. Dispersion relations for extensional and flexural waves for a Varga material: $\lambda_1 = 2.0$, $\lambda_2 = 1.4$, $\mu = 1.7$, $\sigma_2 = 1.6$. Corresponding to $\alpha = 2.0$, $2\beta = 2.8$ and $\gamma = 0.98$. (a) Fundamental modes, (b) first four harmonics.

that as $kh \to \infty$ both q_1 and q_2 are imaginary and that $|q_1| \to |q_2|$. Accordingly, eqn (10) may now be invoked to deduce that the limit of all harmonics is not given by (16) but by

$$|q_1| \to |q_2|, \quad \rho v^2 \to 2\beta - 2\gamma + 2\sqrt{\gamma^2 + \alpha\gamma - 2\gamma\beta}, \tag{18}$$

this being the appropriate limit of all harmonics whenever (17) is violated and therefore case 2 may be classified as

$$\text{Case 2}: 2\beta - \alpha < 0. \tag{19}$$

The limit of eqn $(18)_2$ is the speed of propagation of the shear wave associated with the double root of (10), this wave having wave normal

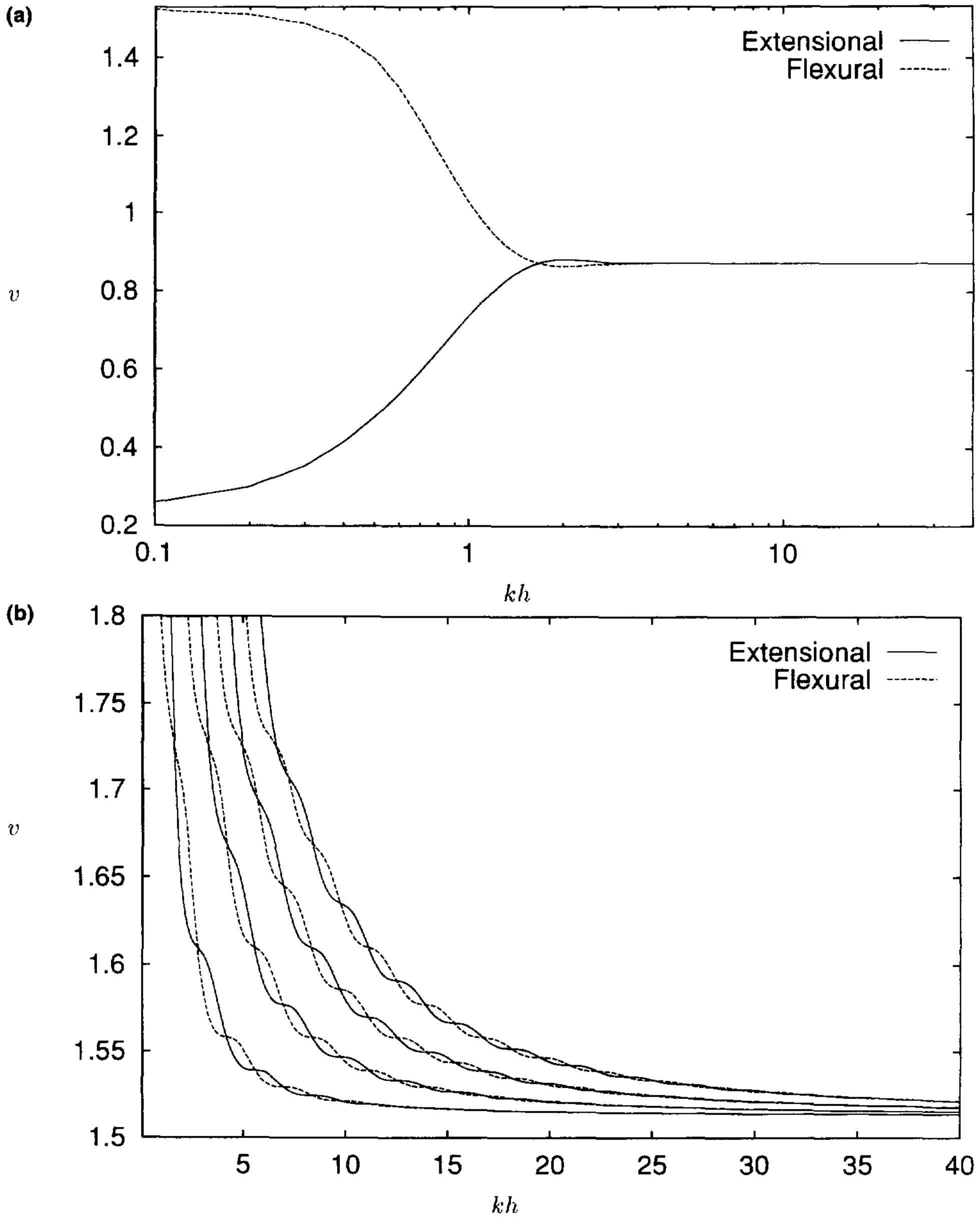

Fig. 2. Dispersion relations for extensional and flexural waves for a Varga material: $\lambda_1 = 1.8$, $\lambda_2 = 0.7$, $\mu = 2.31$, $\sigma_2 = 1.2$. Corresponding to $\alpha = 3.0$, $2\beta = 1.6$ and $\gamma = 0.45$. (a) Fundamental modes, (b) first four harmonics.

$$\mathbf{n} = (1, \pm\varepsilon, 0)(1+\varepsilon^2)^{-1/2}, \quad \varepsilon^2 = \frac{-\gamma + \sqrt{\gamma^2 + \gamma\alpha - 2\beta\gamma}}{\gamma}. \tag{20}$$

A further striking point observed from Fig. 2(b) is that the passage to the limit (17) is accompanied by sinusoidal behaviour. Furthermore, the extensional and flexural curves cross at certain values of wave number. This is something which can only occur when $\rho v^2 < \alpha$, in which case both roots from (10) are negative and therefore yield two imaginary values for q. The fact that the branches cross over may well have significance in the numerical inversion of the transform solutions used to determine transient impact response, see Rogerson (1992). In the specific case of the Varga material it is observed from (15) and (17) that case 1 will occur whenever $2\lambda_2 \geqslant \lambda_1$, with case 2 occurring otherwise. Discussion of a related problem, in the context of the upper bound of Stoneley waves in incompressible layered media, may be found in Ogden and Sotiropoulos (1995). Finally, we remark that a subtle difference for the fundamental modes of (12) and (13) as $kh \to \infty$ is that in case 1

$(16)_2$ form the upper bound of the surface wave speed, whilst in case 2 the upper bound of the subsonic surface wave speed interval is afforded by $(18)_2$.

5. ASYMPTOTIC ANALYSIS OF THE DISPERSION RELATION FOR EXTENSIONAL WAVES

The asymptotic behaviour of both dispersion relations (12) and (13) is dependent on the nature of the solutions of (10) in the appropriate wave number regime. The possible cases which may occur will now be dealt with in turn.

5.1. *Short wavelength limit for the fundamental mode*

When the two roots of eqn (10) are both positive or complex conjugates q_1 and q_2 are either both positive or a conjugate pair. In either case the short wavelength limit ($kh \to \infty$) of eqn (12) is given by

$$\eta^3+\eta^2+\left\{\frac{2\beta-\alpha}{\gamma}+2(1-\sigma)\right\}\eta-(1-\sigma)^2=0, \quad \eta=\sqrt{\frac{\alpha-\rho v^2}{\gamma}}, \quad \sigma=\frac{\sigma_2}{\gamma}. \tag{21}$$

Equation (21) is one from which the speed of any possible Rayleigh surface wave may be obtained. It should be stressed that real values of this wave speed only exist for certain states of pre-stress, for a detailed discussion of this range see Dowaikh and Ogden (1990). Also note that the corresponding limit of (13) is again the surface wave speed eqn (21).

5.2. *Short wavelength limit for the harmonics*

5.2.1. *Case* 1 : ($2\beta \geqslant \alpha$). In the case in which one root of (10) is positive and the other negative we deduce, without loss of generality, that $q_1 = i\hat{q}$, where $\hat{q} \geqslant 0$ is real. It is reiterated that numerical calculation indicate that $\hat{q} \to 0$ as $kh \to \infty$. For the case in which q_1 is imaginary and q_2 real the dispersion relation may be written in the form

$$q_2\{\gamma(1-\hat{q}^2)-\sigma_2\}^2\tan(\hat{q}kh)=\hat{q}\{\gamma(1+q_2^2)-\sigma_2\}^2\tanh(q_2kh). \tag{22}$$

From this equation it is readily deduced that as $kh \to \infty$ and $\hat{q} \to 0$ that $\tan(\hat{q}kh) \sim \hat{q}$. This may be used to deduce that $\hat{q} \approx n\pi/kh$, which upon invoking (11) implies that

$$\rho v^2 \approx \alpha+(2\beta-\alpha)\left(\frac{n\pi}{kh}\right)^2, \quad n=1,2,3\ldots \tag{23}$$

indicating that the limit of all harmonics is in accordance with $(16)_2$. A higher order approximation to this limit in the high wave number regime may be obtained by setting

$$\hat{q}kh=n\pi+\frac{\phi_1}{kh}+O(kh)^{-2}, \quad \tan(\hat{q}kh)=\frac{\phi_1}{kh}+O(kh)^{-2}, \tag{24}$$

where ϕ_1 is to be determined. In addition, eqn $(11)_1$ may be employed to deduce that

$$q_2^2=\frac{2\beta-\alpha}{\gamma}+\left\{\frac{\gamma+\alpha-2\beta}{\gamma}\right\}\left\{\left(\frac{n\pi}{kh}\right)^2+\frac{2n\pi\phi_1}{(kh)^3}\right\}+O(kh)^{-4}. \tag{25}$$

Equations (24) and (25) may now be inserted into eqn (22), and equating leading order terms reveals that

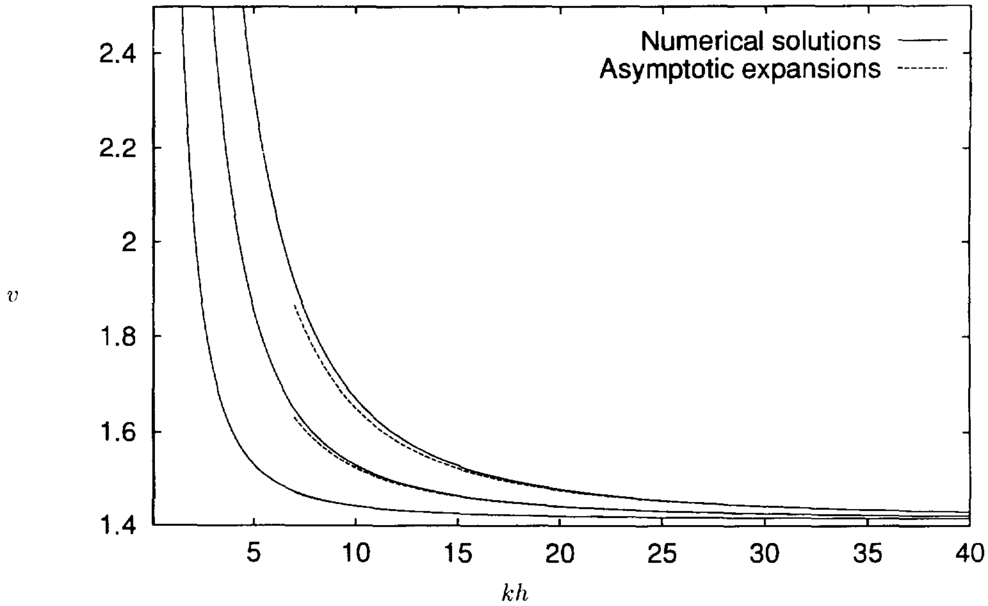

Fig. 3. First three harmonics for extensional waves, showing both numerical and asymptotic solutions (27). The same material parameters are used as in Fig. 1.

$$\phi_1 = n\pi\sqrt{\frac{\gamma}{2\beta-\alpha}}\left(\frac{\gamma+2\beta-\alpha-\sigma_2}{\gamma-\sigma_2}\right)^2, \quad 2\beta \neq \alpha. \tag{26}$$

It should be noted that the value of ϕ_1 is only defined provided $2\beta \neq \alpha$. Indeed, in the neighbourhood of $2\beta = \alpha$ the asymptotic orders must be modified, it is now tacitly assumed that 2β is not equal to α. Even in the case $2\beta = \alpha$ the limit is still given by $\rho v^2 \to \alpha$, however, the higher order correction term is no longer $O(kh)^{-2}$, see eqn (23). Finally, eqn $(11)_1$ may be utilised in conjunction with $(24)_1$ (25) and (26) to conclude that

$$\rho v^2 \approx \alpha+(2\beta-\alpha)\left(\frac{n\pi}{kh}\right)^2\left\{1+\left(\frac{2}{kh}\right)\sqrt{\frac{\gamma}{2\beta-\alpha}}\left(\frac{\gamma+2\beta-\alpha-\sigma_2}{\gamma-\sigma_2}\right)^2\right\}, \quad n = 1,2,3,\ldots, \quad \alpha \neq 2\beta. \tag{27}$$

Equation (27) is a generalisation of the expansion derived for a class of material for which $\alpha+\gamma = 2\beta$, see Rogerson and Fu (1995). Indeed, their results may be obtained easily as a special case of (27).

The asymptotic solutions (27) are shown in Fig. 3. Specifically the first three harmonics are shown for a Varga material with the same material parameters as Fig. 1. For each harmonic the asymptotic solution shows excellent agreement with the numerical solution in the high and moderate wave number regimes. Not surprisingly these solutions are observed to lose accuracy as the harmonic number increases, this being attributable to the occurrence of n in the expansions. If greater accuracy were required one could either obtain higher order terms or seek different expansions appropriate for large harmonic number n, this latter option being carried out later in this section.

5.2.2. *Case* 2: ($2\beta < \alpha$). In this case it is reiterated that numerical results indicate that as $kh \to \infty$ both q_1 and q_2 are purely imaginary, the appropriate form of (12) then being

$$\hat{q}_2\{\gamma(1-\hat{q}_1^2)-\sigma_2\}^2\tan(\hat{q}_1 kh) = \hat{q}_1\{\gamma(1-\hat{q}_2^2)-\sigma_2\}^2\tan(\hat{q}_2 kh), \tag{28}$$

with $q_1 = i\hat{q}_1$ and $q_2 = i\hat{q}_2$, where $\hat{q}_1$, $\hat{q}_2 \geqslant 0$ are real. In view of the fact that $|q_1| \to |q_2|$ as $kh \to \infty$ we assume that

$$q_1^2 = -a+b, \quad q_2^2 = -a-b, \quad b \to 0 \quad \text{as} \quad kh \to \infty \tag{29}$$

within which $a > 0$ and $b \geqslant 0$ are given explicitly by

$$a = \frac{\rho v^2 - 2\beta}{2\gamma}, \quad b = \frac{\sqrt{(2\beta - \rho v^2)^2 - 4\gamma(\alpha - \rho v^2)}}{2\gamma}, \tag{30}$$

use having been made of (10). It is now possible to eliminate ρv^2 from eqns (11) and then after making use of eqns $(29)_{1,2}$ obtain the quadratic equation

$$\gamma a^2 + 2\gamma a + 2\beta - \alpha - \gamma b^2 = 0, \tag{31}$$

which for small b implies that

$$a = a_0 + a_1 b^2 + O(b^4), \quad a_0 = \frac{-\gamma + \sqrt{\gamma^2 + \alpha\gamma - 2\beta\gamma}}{\gamma}, \tag{32}$$

with a_1 defined in terms of a_0 through the relation

$$a_1 = \frac{1}{2(a_0+1)}, \tag{33}$$

and the fact that $a > 0$ has been used to dismiss one root of (31). A combination of (29) and (32) may now be invoked to obtain expansions for $\hat{q}_1$ and $\hat{q}_2$, thus

$$q_1 = i\sqrt{a_0}\left(1 - \frac{b}{2a_0} + \chi b^2 + O(b^3)\right), \quad q_2 = i\sqrt{a_0}\left(1 + \frac{b}{2a} + \chi b^2 + O(b^3)\right), \tag{34}$$

where

$$\chi = \frac{4a_0 a_1 - 1}{8a_0^2}.$$

If terms up to and including $O(b^2)$ in these expansions are inserted into (28) we obtain, after some algebraic manipulation, that

$$\left\{\xi^{(1)} + \frac{\xi^{(0)}}{2a_0}\right\} b \sin(2kh(\sqrt{a_0} + \chi b^2)) = \left\{\xi^{(0)} + \left(\xi^{(2)} + \frac{\xi^{(1)}}{2a_0} + \frac{\xi^{(0)}\chi}{\sqrt{a_0}}\right) b^2\right\} \sin\left(\frac{bkh}{\sqrt{a_0}}\right) + O(b^3), \tag{35}$$

where the $O(1)$ functions $\xi^{(0)}$, $\xi^{(1)}$ and $\xi^{(2)}$ are defined by

$$\xi^{(0)} = \{\gamma(1-a_0) - \sigma_2\}^2, \quad \xi^{(1)} = 2\gamma\{\gamma(1-a_0) - \sigma_2\}, \quad \xi^{(2)} = \gamma^2 - a_1\xi^{(1)}. \tag{36}$$

It is now observed from (35) that as $kh \to \infty$, $b \to 0$, that

$$\sin\left(\frac{bkh}{\sqrt{a}}\right) \sim b, \quad b = \frac{\sqrt{a_0}n\pi}{kh} + O(kh)^{-2}. \tag{37}$$

Equation $(11)_1$ may now be used in conjunction with $(37)_2$ and $(32)_1$ to obtain an approximation for ρv^2, thus

$$\rho v^2 \approx 2\beta + 2\gamma a_0 + \left(\frac{\gamma a_0}{1+a_0}\right)\left(\frac{n\pi}{kh}\right)^2. \tag{38}$$

It is possible to improve the approximation (38) by putting

$$b = \frac{\sqrt{a_0}n\pi}{kh} + \frac{\phi_2}{(kh)^2} + O(kh)^{-3}, \tag{39}$$

which may be used to establish that

$$\sin(2(\sqrt{a_0} + \chi b^2)kh) = \sin(2\sqrt{a_0}kh) + 2a_0\chi\frac{n^2\pi^2}{kh}\cos(2\sqrt{a_0}kh) + O(kh)^{-2}, \tag{40}$$

$$\sin\left(\frac{bkh}{\sqrt{a_0}}\right) = \frac{(-1)^n\phi_2}{\sqrt{a_0}kh} + O(kh)^{-2}. \tag{41}$$

The expansions (39)–(41) may now be inserted into (35), and upon equating like powers of kh we conclude that

$$\phi_2 = \frac{(-1)^n a_0}{\xi^{(0)}}\left\{\xi^{(1)} + \frac{\xi^{(0)}}{2a_0}\right\}n\pi\sin(2\sqrt{a_0}kh), \tag{42}$$

which upon invoking eqns (39) and $(11)_1$ enables us to obtain the following approximation for ρv^2

$$\rho v^2 \approx 2\beta + 2\gamma a_0 + \left(\frac{\gamma a_0}{1+a_0}\right)\left(\frac{n\pi}{kh}\right)^2\left\{1 + \frac{(-1)^n}{kh}\mathscr{A}\sin(2\sqrt{a_0}kh)\right\}, \tag{43}$$

with

$$\mathscr{A} = \frac{2\sqrt{a_0}}{\xi^{(0)}}\left\{\xi^{(1)} + \frac{\xi^{(0)}}{2a_0}\right\} \quad \text{and} \quad n = 1, 2, 3, \ldots. \tag{44}$$

With use of eqn $(32)_2$ it is readily established that all harmonics tend to the wave speed given explicitly by $(18)_2$. However, in contrast to the previous case the passage to this limit is accompanied by a sinusoidal contribution at third order.

In Fig. 4 the asymptotic approximations derived in this section are shown with the corresponding numerical solution. The second order approximation (38) is shown in Fig. 4(a), with the higher order approximation (43) depicted in Fig. 4(b). The improvement, certainly in the high wave number regime, is apparent and the third order approximation yields excellent approximations to the numerical solutions. In particular, the inclusion of the third order sinusoidal term enables the asymptotic solution to follow the numerical solution far more closely. The loss in accuracy as the harmonic number increases is again evident.

5.3. *Long wave limit* kh $\to 0$

As $kh \to 0$ previous numerical calculations indicated that only the fundamental mode of (12) retains $O(1)$ wave speed, this limit being given explicitly by

$$\rho v^2 = 2\beta + 2\gamma - 2\sigma_2. \tag{45}$$

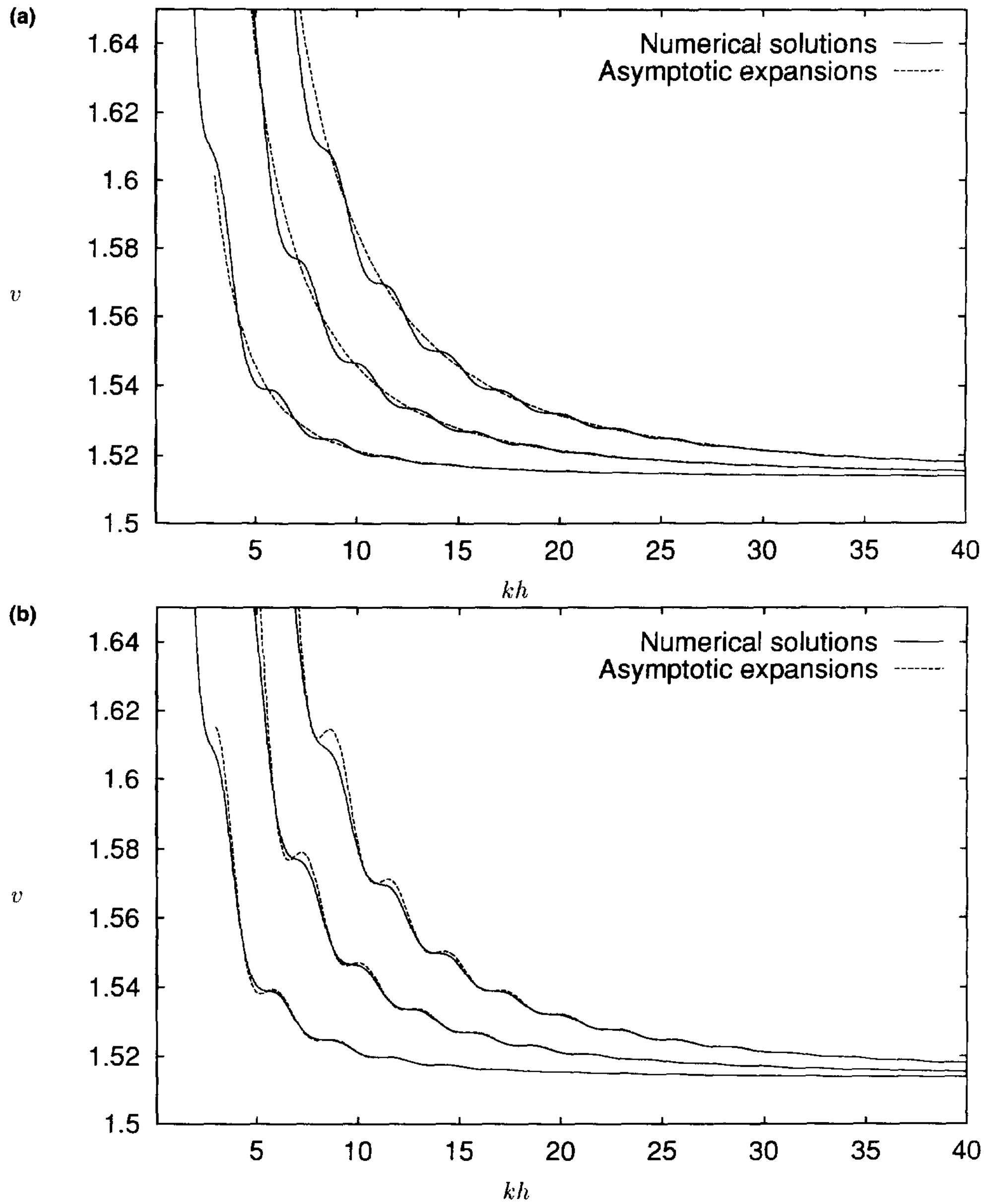

Fig. 4. First three harmonics for extensional waves, showing both numerical and asymptotic solutions. The same material parameters are used as in Fig. 1, (a) second order asymptotic expansion (38), (b) third order asymptotic expansion (43).

For all harmonics it is therefore assumed that ρv^2 is large, and thus from (10) it is established that as $kh \to 0$

$$q_1^2 = -\frac{\rho v^2}{\gamma} + \frac{2\beta-\gamma}{\gamma} + \frac{\alpha+\gamma-2\beta}{\rho v^2} + O\left(\frac{1}{\rho v^2}\right)^2, \tag{46}$$

$$q_2^2 = 1 + \frac{2\beta-\alpha-\gamma}{\rho v^2} + O\left(\frac{1}{\rho v^2}\right)^2. \tag{47}$$

It is therefore assumed that $q_1 = i\hat{q}$, with $\hat{q} \geqslant 0$ and then eqn (12) implies that

$$\tan(kh\hat{q}) \sim kh\hat{q}^{-3}, \quad kh\hat{q} = n\pi + \phi_3 (kh)^m + \cdots, \tag{48}$$

where ϕ_3 is to be determined and $(48)_1$ and $(48)_2$ implies that, provided $n \neq 0$, $m = 4$. The expansions (48) are now inserted into (12) to obtain

$$\phi_3 = \frac{(\gamma(1+q_2^2)-\sigma_2)^2}{\gamma^2 (n\pi)^3}. \tag{49}$$

A combination of eqns (11), (47) and (48) may now be used to establish the approximation

$$\rho v^2 \approx \gamma\left(\frac{n\pi}{kh}\right)^2 + 2\beta - \gamma + \left\{\frac{2(2\gamma-\sigma_2)^2 + 2\beta - \alpha - \gamma}{\gamma}\right\}\left(\frac{kh}{n\pi}\right)^2, \tag{50}$$

thus indicating that for all harmonics $\rho v^2 \to \infty$ as $kh \to 0$.

5.4. *Large harmonic number expansion*

It has already been remarked that the approximations (27) and (43) remain valid only when the harmonic number n is not comparable in magnitude with kh. In this section asymptotic expansions appropriate for large n and moderate kh will be sought. For these harmonics when kh is $O(1)$, $\rho v^2 \gg 1$ and there is then obviously no need to distinguish between cases 1 and 2, the appropriate form of dispersion relation now being given by (22). Making use of the expansions for q_1^2 and q_2^2 given in (46) and (47) it is inferred that if $\hat{q} \gg 1$ and kh is $O(1)$, then $\tan(\hat{q}kh) \sim \hat{q}^{-3}$, and therefore

$$\hat{q}kh = n\pi + \frac{\hat{\phi}_1}{n^3} + \cdots, \quad \tan(\hat{q}kh) = \frac{\hat{\phi}_1}{n^3} + \cdots. \tag{51}$$

These two expansions may now be used with eqn (22) to deduce that

$$\hat{\phi}_1 = \frac{(kh)^3 \{\gamma(1-q_2^2)-\sigma_2\}^2 \tanh(q_2 kh)}{\pi^3 \gamma^2 q_2}, \tag{52}$$

which upon also making use of eqns $(11)_1$, (47) and $(51)_1$ yields the approximation

$$\rho v^2 \approx \gamma\left\{\frac{n\pi}{kh}\right\}^2 + 2\beta - \gamma + \frac{kh}{\gamma(n\pi)^2}\{2(2\gamma-\sigma_2)^2 \tanh(kh) + \gamma(\alpha+\gamma-2\beta)kh\}. \tag{53}$$

In Fig. 5 the approximations (53) are shown with the numerical solution for the third, fourth and fifth harmonics. These clearly indicate that, as expected, the approximation for the lower harmonics becomes worse as kh increases. Notwithstanding this, these approximations provide a good representation of these harmonics in the moderate wave number regime.

6. ASYMPTOTIC ANALYSIS OF THE DISPERSION RELATION FOR FLEXURAL WAVES

When the two roots of (10) are either real or complex conjugates as $kh \to \infty$ the limit of the fundamental mode of (13) is identical to that for extensional waves, see eqn (21). Although the various limits associated with the harmonics are the same as those in Section 5 the asymptotic expansions are different from those previously derived. The various possible cases will now be discussed and the resulting asymptotic expansion derived.

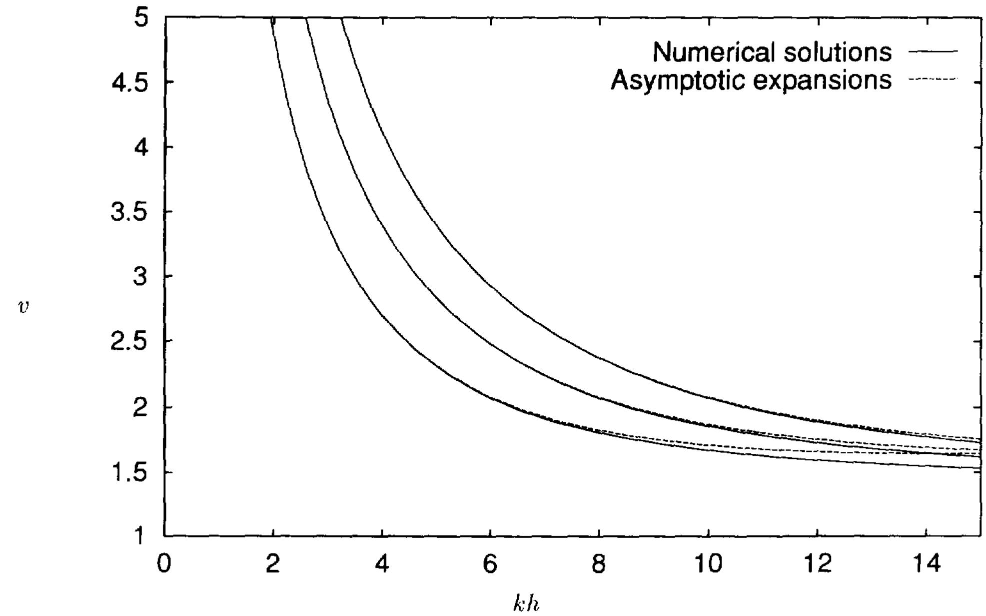

Fig. 5. Third, fourth and fifth harmonics for extensional waves showing both the numerical solution and the high harmonic number approximation (53). The same material parameters are used as in Fig. 1.

6.1. *Short wave limit for the harmonics*

6.1.1. *Case* 1 : ($2\beta \geqslant \alpha$). Using similar arguments to those already employed in the corresponding limit of (12), it is assumed that $q_1 = i\hat{q}$ and that as $kh \to \infty$, $\hat{q} \to 0$. The dispersion relation (13) may now be cast into the appropriate form

$$q_2\{\gamma(1-\hat{q}^2)-\sigma_2\}^2 \tanh(kq_2h) = -\hat{q}\{\gamma(1+q_2^2)-\sigma_2\}^2 \tan(\hat{q}kh). \tag{54}$$

From this equation it is evident that $\tan(\hat{q}kh) \sim \hat{q}^{-1}$, indicating that $kh\hat{q} \approx (n+1/2)\pi$, which when employed in conjunction with (11) yields the approximation

$$\rho v^2 \approx \alpha + (2\beta-\alpha)\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{(kh)^2}. \tag{55}$$

Equation (55) indicates that for all harmonics as $kh \to \infty$, $\rho v^2 \to \alpha$, as also indicated by (23). An improved approximation may be sought by assuming that

$$\hat{q}kh = \left(n+\frac{1}{2}\right)\pi + \frac{\phi_4}{kh} + O(kh)^{-2}, \quad \tan(\hat{q}kh) = -\frac{kh}{\phi_4} + O(1), \tag{56}$$

and upon invoking (11) that

$$q_2^2 = \frac{2\beta-\alpha}{\gamma} + \left\{\frac{\gamma+\alpha-2\beta}{\gamma}\right\}\left\{\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{(kh)^2} + \frac{2(n+\frac{1}{2})\pi\phi_4}{(kh)^3} + O(kh)^{-4}\right\}. \tag{57}$$

It is now possible to use expansions (56) and (57) to deduce that

$$\phi_4 = \left(n+\frac{1}{2}\right)\pi\sqrt{\frac{\gamma}{2\beta-\alpha}}\left(\frac{\gamma+2\beta-\alpha-\sigma_2}{\gamma-\sigma_2}\right)^2, \quad \alpha \neq 2\beta. \tag{58}$$

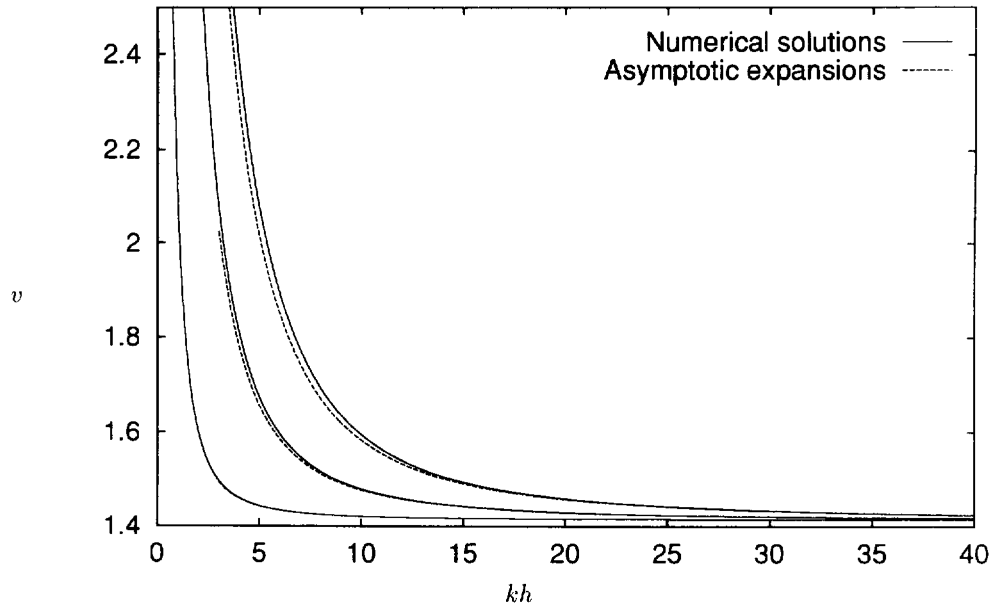

Fig. 6. First three harmonics for flexural waves, showing both numerical and asymptotics solutions (59). The same material parameters are used as in Fig. 2.

Equations (56), (54) and (58) may now be invoked with eqn $(11)_1$ to establish that

$$\rho v^2 \approx \alpha+(2\beta-\alpha)\left(n+\frac{1}{2}\right)^2\frac{\pi^2}{(kh)^2}\left\{1+\frac{2}{kh}\sqrt{\frac{\gamma}{2\beta-\alpha}}\left(\frac{\gamma+2\beta-\alpha-\sigma_2}{\gamma-\sigma_2}\right)^2\right\}, n=0,1,2,\ldots, \tag{59}$$

indicating that the limit of all harmonics is now given by $(16)_2$.

A glance at the expansion (27) reveals that the only difference between that and (59) is that n is now replaced by $n+1/2$. The implication is that within the high wave number regime extensional and flexural branches interlace each other, i.e., they never intersect. The first three harmonics of flexural motions are shown in Fig. 6, this figure showing both numerical and asymptotic solutions. Remarkable agreement is observed between the asymptotic and numerical solution with, in particular, the asymptotic solution virtually indistinguishable from the numerical solution for the first harmonic.

6.1.2. *Case* 2: $(2\beta<\alpha)$. In this case the analogous form of (28) associated with (13) is given by

$$\hat{q}_2\{\gamma(1-\hat{q}_1^2)-\sigma_2\}^2\tan(\hat{q}_2kh) = \hat{q}_1\{\gamma(1-\hat{q}_2^2)-\sigma_2\}^2\tan(\hat{q}_1kh). \tag{60}$$

If use is now made of eqn (34) this dispersion relation may be approximated in the high wave number regime by

$$\left\{\xi^{(1)}+\frac{\xi^{(0)}}{2a_0}\right\}b\sin(2kh(\sqrt{a_0}+\chi b^2)) = -\left\{\xi^{(0)}+\left(\xi^{(2)}+\frac{\xi^{(1)}}{2a_0}+\frac{\xi^{(0)}\chi}{\sqrt{a_0}}\right)b^2\right\}\sin\left(\frac{bkh}{\sqrt{a_0}}\right)+O(b^3), \tag{61}$$

with $\xi^{(i)}$, $i\in\{1,2,3\}$ defined in eqn (36). It is evident from (61) that $\sin(kbh/\sqrt{a})\sim b$, and therefore an approximation up to $O(kh)^{-2}$ is exactly that shown in eqn (38). Utilisation of

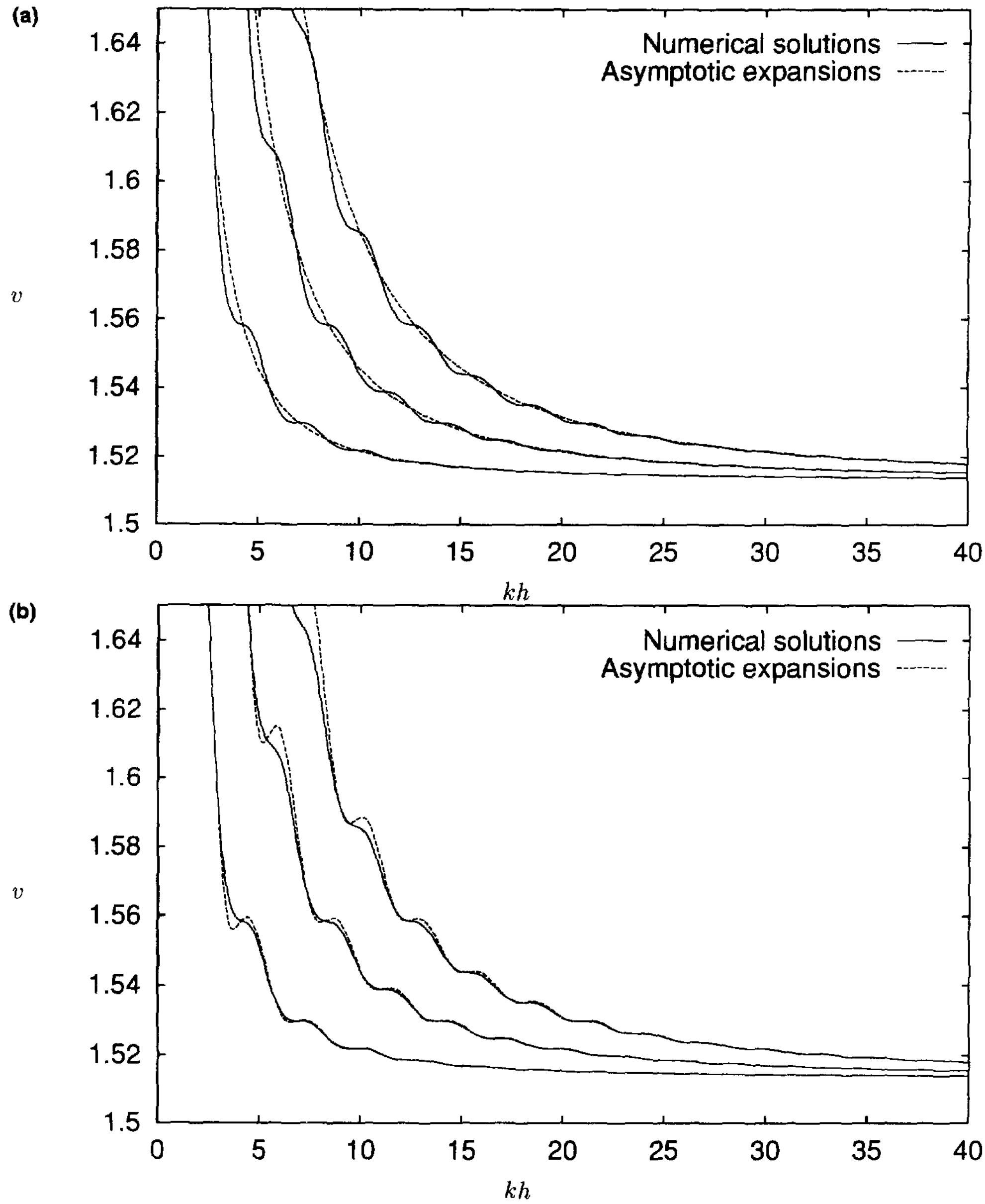

Fig. 7. First three harmonics for flexural waves, showing both numerical and asymptotics solutions. The same material parameters are used as in Fig. 2, (a) second order asymptotic expansion (38), (b) third order asymptotic expansion (62).

eqns (39)–(41) may now be used within eqn (61) to yield a higher order approximation to b. This is then used to establish that

$$\rho v^2 \approx 2\beta + 2\gamma a_0 + \left(\frac{a_0 \gamma}{1+a_0}\right)\left(\frac{n\pi}{kh}\right)^2 \left\{1 + \frac{(-1)^{n+1}}{kh} \mathscr{A} \sin(2\sqrt{a_0}kh)\right\}. \tag{62}$$

It is observed that the only difference between the asymptotic expansions (62) and (43) is that $(-1)^n$ has been replaced by $(-1)^{n+1}$. The implication within the high wave number regime is that the n^{th} harmonics of both flexural and extensional motions continually cross over. This contrasts with case 1 in which adjacent harmonics never intersect and extensional and flexural branches interlace. The second order approximation (38) is shown in Fig. 7(a) with the improved higher order approximation (62) shown in Fig. 7(b). It has already been stated that the second order approximation for flexural waves is identical to the corresponding extensional wave expansion. The implication is that the third order expansion provides a significant improvement which fully captures the oscillatory behaviour. Indeed the improvement in using the third order expansion is obvious if you compare Figs 7(a) and 7(b), especially in the high wave number regime.

6.2. *Long wavelength limit* kh $\to 0$

In the limit $kh \to 0$ the finite limiting wave speed of the fundamental mode of (13) is given by

$$\rho v^2 = \alpha - \frac{(\gamma-\sigma_2)^2}{\gamma}. \tag{63}$$

In seeking appropriate expansions for the harmonics the two expansions (46) and (47) may be invoked, in conjunction with (13), to establish that $\tan(\hat{q}kh) \sim \hat{q}^3 kh$ and therefore

$$|\hat{q}|kh = \left(n+\frac{1}{2}\right)\pi + \phi_6 (kh)^2 + O(kh)^3, \quad \tan(\hat{q}kh) = -\frac{(kh)^{-2}}{\phi_6} + O(1). \tag{64}$$

It is now possible to deduce that

$$\phi_6 = \frac{\{\gamma(1+q_2^2)-\sigma_2\}^2}{\gamma^2 q_2^2 (n+\frac{1}{2})^3 \pi^3}. \tag{65}$$

Finally, eqn (65) may be inserted into $(64)_1$ and use made of $(11)_1$ and (47) to deduce that

$$\rho v^2 \approx \frac{\gamma(n+\frac{1}{2})^2\pi^2}{(kh)^2} + 2\beta - \gamma + \left\{\frac{\{2(2\gamma-\sigma_2)^2 + 2\beta - \alpha - \gamma\}(kh)^2}{\gamma(n+\frac{1}{2})^2\pi^2}\right\}. \tag{66}$$

6.3. *Large harmonic number*

In a similar way to the corresponding case for extensional waves, it is deduced from (54) that when n is large $\tan(\hat{q}kh) \sim \hat{q}^3$, and so

$$\hat{q}kh = \left(n+\frac{1}{2}\right)\pi + \frac{\hat{\phi}_2}{(n+\frac{1}{2})^3} + \cdots, \quad \tan(\hat{q}kh) = -\frac{(n+\frac{1}{2})^3}{\hat{\phi}_2} + \cdots. \tag{67}$$

Upon inserting these expansions into (54) it is readily established that

$$\hat{\phi}_2 = \frac{(kh)^3 \{\gamma(1+q_2^2)-\sigma_2\}^2}{q_2 \gamma^2 \pi^3 \tanh(q_2 kh)}, \tag{68}$$

which upon utilising eqns $(11)_1$ and (47) yields the appropriate approximation

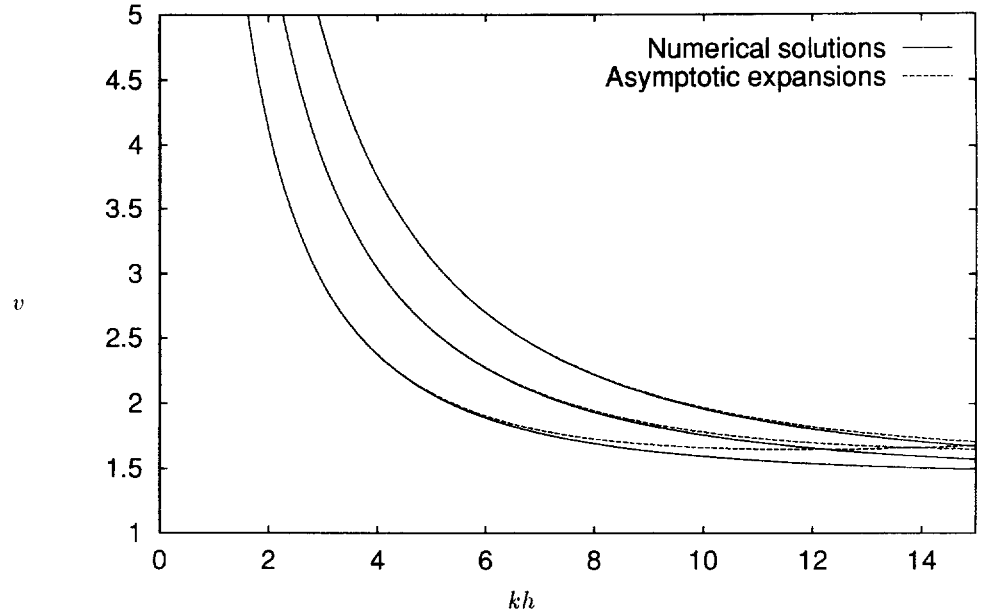

Fig. 8. Third, fourth and fifth harmonics for extensional waves showing both the numerical solution and the high harmonic number approximation (69). The same material parameters are used as in Fig. 2.

$$\rho v^2 \approx \frac{\gamma(n+\frac{1}{2})^2\pi^2}{(kh)^2} + 2\beta - \gamma + \frac{kh}{\gamma(n+\frac{1}{2})^2\pi^2}\left\{\frac{2(2\gamma-\sigma_2)^2}{\tanh(kh)} + \gamma kh(\alpha+\gamma-2\beta)\right\}. \quad (69)$$

In Fig. 8 the approximations (69) are shown with the corresponding numerical solutions. These again indicate that the approximations are better at a fixed wave number as the harmonic number increases, however, offering a surprisingly good representation for such low harmonics.

REFERENCES

Dowaikh, M. A. and Ogden, R. W. (1990) On surface waves and deformations in a pre-stressed incompressible elastic solid. *IMA Journal of Applied Mathematics* **44,** 261–284.

Ogden, R. W. (1984) *Non-linear Elastic Deformations.* Ellis Horwood, New York.

Ogden, R. W. and Roxburgh, D. G. (1993) The effect of pre-stress on the vibration and stability of elastic plates. *International Journal of Engineering Science* **30,** 1611–1639.

Ogden, R. W. and Sotiropoulos, D. A. (1995) On interfacial waves in pre-stressed layered incompressible solids. *Proceedings of the Royal Society London* **A450,** 319–341.

Rogerson, G. A. (1992) Penetration of impact waves in a six-ply fibre composite laminate. *Journal of Sound and Vibration* **158,** 105–120.

Rogerson, G. A. and Fu, Y. B. (1995) An asymptotic analysis of the dispersion relation of a pre-stressed incompressible elastic plate. *Acta Mechanica* **111,** 59–77.

Rogerson, G. A. and Sandiford, K. J. (1996) On small amplitude vibrations of pre-stressed laminates. *International Journal of Engineering Science* **34**(8), 853–872.

Sheridan, P. M., James, F. O. and Miller, T. S. (1992) Design of components. In *Engineering With Rubber* (ed. A. N. Gent), pp. 209–235. Hanser, Munich.